① Solve the following equation

$$\begin{cases} -\Delta u + u = f(x,y), & (x,y) \in \Omega = [0,1]\times[0,1] \\ u\big|_{\partial\Omega} = g(x,y) \end{cases}$$

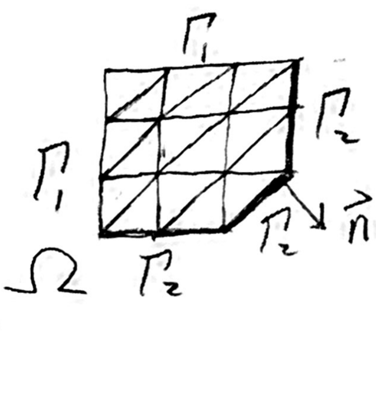
on a rectangular grid with the 5-point difference scheme (with $\Delta x = \Delta y = h$)

$$-\left\{ \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{h^2} + \frac{U_{i,j+1} - 2U_{ij} + U_{i,j-1}}{h^2} \right\} + U_{ij} = f_{ij}.$$

Design a multi-grid method to solve the problem with optimal complexity. Describe your method clearly.

② Solve $\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u\big|_{\Gamma_1} = g_1, & \dfrac{\partial u}{\partial \vec{n}}\Big|_{\Gamma_2} = g_2 \end{cases}$

with a piecewise linear continuous FEM. Describe your method clearly and write down the resulted linear system with every entry of the matrix and vectors clearly defined, indicating the zeros. Show that the linear system has a unique solution.



Here $\vec{n}$ is the outward pointing unit normal vector. Use the provided mesh for your FEM.

③ Write down the Crank-Nicolson scheme for solving $u_t + au_x + bu_y = 0$ (a, b are constants). Transform the scheme into an ADI scheme and study its accuracy and stability.

④ Consider the equation $u_t + au_x = 0$ where "a" is a constant. Verify that the center difference scheme (with forward Euler time discretization) has a Fourier symbol whose magnitude is ≤ 2 if the CFL number is $\leq \sqrt{3}$. Therefore it can be stabilized by BFECC with CFL number $\leq \sqrt{3}$.

⑤ Show that the semi-discrete MUSCL scheme is 2nd order accurate for approximating the smooth solution u of the conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0,$$

where f is a smooth function. To simplify your analysis, the "minmod" function in the MUSCL scheme is replaced by the average of its arguments and the flux function $F(\cdot,\cdot)$ in the scheme is assumed to be the Lax-Friedrichs flux.