H.W. #1

Prob. 1 Study the truncation error and the stability for the following schemes:

Equ: $u_t + a u_x = \nu u_{xx}$, where a, $\nu > 0$ are constants.

Scheme:

(1) $$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a \frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} = \nu \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Delta x^2}.$$

(2) $$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a \cdot \frac{1}{2}\left[\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2\Delta x} + \frac{U_{j+1}^{n} - U_{j-1}^{n}}{2\Delta x}\right]$$
$$= \nu \cdot \frac{1}{2}\left[\frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{\Delta x^2} + \frac{U_{j+1}^{n} - 2U_j^{n} + U_{j-1}^{n}}{\Delta x^2}\right].$$

(3) $$\frac{U_j^{n+1} - U_j^n}{\Delta t} + a \frac{U_j^n - U_{j-1}^n}{\Delta x} = \nu \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2},$$

here $a > 0$.

Prob. 2 Compute $$\begin{cases} u_t + a u_x = \nu u_{xx}, & t > 0,\ x \in \mathbb{R} \\ u(x,0) = \sin(2\pi x), & x \in \mathbb{R} \end{cases}$$

with (1) $a = 1$, $\nu = 0.1$; (2) $a = 0$, $\nu = 1$.

Find the solution at $T = 0.1$ and 0.5 (final time) with any scheme you wish and with different mesh sizes to see if your numerical results converge or not.

Hint: you may choose one period of the initial data as the computation interval and apply the periodic boundary condition.

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Prob. 3

For $u_t - u_{xx} = 0$

The implicit scheme can be written as

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} - \frac{U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}}{\Delta x^2} = 0$$

show that $|U_i^n - u_i^n| = O(\Delta t + \Delta x^2)$ as $\Delta t, \Delta x \to 0$, $n\Delta t \le T$.

Prob. 4 Suppose a scheme can be written as

$$\sum_{j=-m_1}^{m_1} \alpha_j U_{i+j}^{n+1} = \sum_{j=-m_2}^{m_2} \beta_j U_{i+j}^{n} .$$

Find a set of conditions for $\{\alpha_j, \beta_j\}$ so that it's stable in l^∞, and prove your claim.