Chapter 1

CENTRAL SCHEMES AND CENTRAL DISCONTINUOUS GALERKIN METHODS ON OVERLAPPING CELLS

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Abstract The central scheme of Nessyahu and Tadmor (J. Comput. Phys, 87(1990)) has the benefit of not having to deal with the solution within the Riemann fan for solving hyperbolic conservation laws and related equations. But the staggered averaging causes large dissipation when the time step size is small comparing to the mesh size. The recent work of Kurganov and Tadmor (J. Comput. Phys, 160(2000)) overcomes the problem by use of a variable control volume and obtains a semi-discrete non-staggered central scheme. Motivated by this work, we introduce overlapping cell averages of the solution at the same discrete time level, and develop a simple alternative technique to control the $O(1/\Delta t)$ dependence of the dissipation. Semi-discrete form of the central scheme can also be obtained. This technique is essentially independent of the reconstruction and the shape of the mesh, thus could also be useful for Voronoi mesh. The overlapping cell representation of the solution also opens new possibilities for reconstructions. Generally more compact reconstruction can be achieved. We demonstrate through numerical examples that combining two classes of the overlapping cells in the reconstruction can achieve higher resolution. Overlapping cells create self similarity in the grid and enable the development of central type discontinuous Galerkin methods for convection diffusion equations and elliptic equations with convection, following the series works of Cockburn and Shu (Math. Comp. 52(1989)).

Keywords: Central Scheme, discontinuous Galerkin Method, ENO scheme, MUSCL scheme, TVD scheme.

1. Introduction

Godunov scheme first captures the shock wave in a narrow transition layer. It is based on evolving piece-wise cell average representations of the solution by evaluating the flux at the cell boundary which is obtained from solving a Riemann problem. Various higher resolution schemes has been developed such as FCT, MUSCL, TVD schemes, PPM, ENO, WENO, *etc.* Unlike Godunov scheme, Lax-Friedrich scheme does not need to solve a Riemann problem. The central scheme of Nessyahu and Tadmor (NT) (NeTa90) provides the higher order generalization of the Lax-Friedrich scheme and is based on a staggered average of the piece-wise polynomial representation of the solution, thus avoids dealing with the Riemann fan originated from the jump values at the cell edges. Further developments on central schemes can be found in e.g. SaWe92; JiTa98; JiLeLiOsTa98; LiTa98; BiPuRu99; KuLe00; AeSt03; KuNoPe01; KuTa00; LePuRu02, etc. The relaxation scheme of Jin and Xin (JiXi95) provides another approach to nonlinear conservation laws.

Central schemes provide a black box type solution to nonlinear hyperbolic conservation laws and other closely related equations since essentially one only needs to supply the flux function. Similar approaches can also be achieved with upwind schemes with a Lax-Friedrich type flux function or building block, see e.g. Shu and Osher (ShOs88; ShOs89), Liu and Osher (LiOs98). Since the central schemes usually use staggered average, the time step size cannot be passed to zero. Similar situation occurs in the 2D conservative front tracking and is overcome by use of space-time cells in Glimm *et. al.* (GlLiLiXuZh03). In KuTa00, Kurganov and Tadmor introduce a new kind of central scheme without the large dissipation error related to the small time step size by use of a variable control volume whose size depends on time step size. By passing to the limit as the time step size goes to zero, the non-staggered semi-discrete central Godunov type scheme can be developed to which standard Runge-Kutta methods or the TVD Runge-Kutta methods (Shu and Osher, ShOs88) can be applied. This allows the central scheme to be used for a larger class of equations where time step size could be small comparing to the mesh size.

In Liu (Li04), an alternative technique is introduced to control the dissipative error of central schemes. The major idea is to introduce an overlapping cell representation of the solution which allows a convex combination of the overlapping cell averages. An immediate advantage is that the time discretization becomes simple and more robust by use of the TVD Runge-Kutta method (ShOs88) due to the self similarity of overlapping cells over time. Also by use of a time step size dependent convex combination of the overlapping cell averages, the $O(1/\Delta t)$ dependent dissipative error can be easily controlled. Various reconstruction methods (e.g., MUSCL, ENO, WENO etc) can be applied to the overlapping cells in a standard way by separating them into two classes and applying the reconstruction method to each class. More efficient application of the reconstruction methods using the combined information from the overlapping cell averages has also been explored and require further study particularly in higher space dimensions. The use of overlapping cells opens many new possibilities. For example, central discontinuous Galerkin type approach on overlapping cells becomes feasible due to the self similarity of the cells, following the works of Cockburn and Shu (CoSh89; CoSh91; CoSh98, *etc*). Also the semi-discrete form on overlapping cells results in a central type locally conservative elliptic solver which could be suitable for elliptic equations with large advection.

2. Central Schemes on Overlapping Cells

Consider 1D conservation law

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad (x,t) \in \mathcal{R} \times (0,T). \tag{1.1}$$

Let $\{x_i\}$ be a uniform partition in $\mathcal{R}$, with $\Delta x = x_{i+1} - x_i$. Denote $x_{i+1/2} = \frac{1}{2}(x_i + x_{i+1})$. Let U_i approximate the cell average $\int_{x_{i-1/2}}^{x_{i+1/2}} u(x,t)dx$ and $U_{i+1/2}$ approximate the cell average $\int_{x_i}^{x_{i+1}} u(x,t)dx$. Denote $U_i^n = U_i(t_n)$, $U_{i+1/2}^n = U_{i+1/2}(t_n)$. By applying a MUSCL or ENO reconstruction for the two sets of cell averages, one obtains a function $\mu^n(x)$ which is a piece-wise polynomial for cells $\{(x_{i-1/2}, x_{i+1/2}) : i = 0, \pm1, \pm2, \cdots\}$ and a function $\nu^n(x)$ which is a piece-wise polynomial for cells $\{(x_i, x_{i+1}) : i = 0, \pm1, \pm2, \cdots\}$. For conservation purpose, they should satisfy $\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \mu^n(x)dx = U_i^n$ and $\frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} \nu^n(x)dx = U_{i+1/2}^n$. Let $\Delta t_n = t_{n+1} - t_n$ be the current time step size, following Nessyahu and Tadmor (NeTa90), the central scheme with forward Euler time discretization can be written on overlapping cells as follows

$$\begin{array}{rcl} U_i^{n+1} & = & \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \nu^n(x)dx - \frac{\Delta t_n}{\Delta x}[f(\nu^n(x_{i+1/2})) - f(\nu^n(x_{i-1/2}))], \\ U_{i+1/2}^{n+1} & = & \frac{1}{\Delta x}\int_{x_i}^{x_{i+1}} \mu^n(x)dx - \frac{\Delta t_n}{\Delta x}[f(\mu^n(x_{i+1})) - f(\mu^n(x_i))]. \end{array} \tag{1.2}$$

The higher order time discretization can be obtained by applying the TVD Runge-Kutta time discretization procedure (ShOs88). Kurganov and Tadmor (KuTa00) point out that since the numerical dissipation from $\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \nu^n(x)dx$ does not depend on Δt_n, the cumulative error will

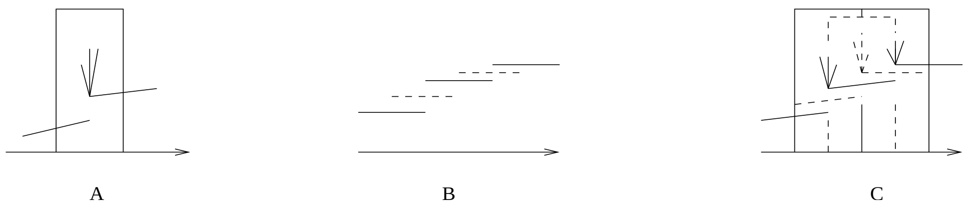

Figure 1.1. (A) NT scheme; (B) 1D overlapping cells; (C) overlapping cells create self similarity for the grid over time and allow a convex combination of the overlapping cell averages to control the dissipation.

depend on $O(1/\Delta t)$, the total number of time steps in the computation. Therefore when Δt is very small, e.g. $\Delta t = O(\Delta x^2)$ for convection diffusion equations, the numerical dissipation becomes large. This is easily seen if $f(u) \equiv 0$, then what the central scheme does is conservative rezoning at every time step, which will smear out the solution with the number of iterations increasing. By choosing the size of the control volume (x_i, x_{i+1}) proportional to Δt, this $O(1/\Delta t)$ dependence can be removed and by passing to the limit as $\Delta t \to 0$, semi-discrete Godunov type central schemes can be developed (KuTa00). Liu (Li04) introduces another easy modification of the NT scheme to remove the $O(1/\Delta t)$ dependence of the error taking advantage of the overlapping cell representation U_i^n and $U_{i+1/2}^n$. The idea is to use a time dependent weighted average of $\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \nu^n(x)dx$ and U_i^n in (1.2), which does not change the order of accuracy of the scheme. In fact the difference between them is the local dissipation error. Suppose $\Delta t_n \le \Delta\tau_n$ and $\Delta\tau_n$ is an upper bound for the current time step size due to the CFL restriction. The forward Euler form of the new central scheme can be formulated as follows

$$\begin{aligned} U_i^{n+1} &= \theta\left(\tfrac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \nu^n(x)dx\right) + (1-\theta)U_i^n - \tfrac{\Delta t_n}{\Delta x}[f(\nu^n(x_{i+1/2})) - f(\nu^n(x_{i-1/2}))], \\ U_{i+1/2}^{n+1} &= \theta\left(\tfrac{1}{\Delta x}\int_{x_i}^{x_{i+1}} \mu^n(x)dx\right) + (1-\theta)U_{i+1/2}^n - \tfrac{\Delta t_n}{\Delta x}[f(\mu^n(x_{i+1})) - f(\mu^n(x_i))], \end{aligned} \tag{1.3}$$

where $\theta = \Delta t_n/\Delta\tau_n$. Note that when $\theta = 1$, it becomes the scheme (1.2). The comparison of schemes (1.2) and (1.3) for Burgers equation with very small time step size can be found in Fig. 1.2 (a) and (b). One can also obtain the following semi-discrete form by moving U_i^n and $U_{i+1/2}^n$ to the left hand side and multiplying both side by $\frac{1}{\Delta t_n}$, then passing to the limit as $\Delta t_n \to 0$

$$\begin{aligned} \tfrac{d}{dt}U_i(t_n) &= \tfrac{1}{\Delta\tau_n\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \nu^n(x)dx - \tfrac{1}{\Delta\tau_n}U_i^n - \tfrac{1}{\Delta x}[f(\nu^n(x_{i+1/2})) - f(\nu^n(x_{i-1/2}))], \\ \tfrac{d}{dt}U_{i+1/2}(t_n) &= \tfrac{1}{\Delta\tau_n\Delta x}\int_{x_i}^{x_{i+1}} \mu^n(x)dx - \tfrac{1}{\Delta\tau_n}U_{i+1/2}^n - \tfrac{1}{\Delta x}[f(\mu^n(x_{i+1})) - f(\mu^n(x_i))]. \end{aligned} \tag{1.4}$$

Note that this semi-discrete form doesn't need to explicitly evaluate the jump values of $\nu^n(x)$ and $\mu^n(x)$ across their respective cell edges (which is one of the features of the NT scheme). See Fig. 1.1. We have the following theorem.

Theorem 1 *Let the schemes (1.2) and (1.3) start from the same time t_n with the same initial values U_i^n, $U_{i+1/2}^n$. If the scheme (1.2) is TVD from time step t_n to $t_n + \Delta\tau_n$, then the scheme (1.3) is also TVD from time t_n to $t_n + \Delta t_n$, for any $\Delta t_n \in [0, \Delta\tau_n]$.*

The theorem provides some insights into two reconstruction procedures: one is standard to reconstruct for the two classes of cell averages $\{U_i^n : i = 0, \pm1, \pm2, \cdots\}$ and $\{U_{i+1/2}^n : i = 0, \pm1, \pm2, \cdots\}$ separately; the other mixes the two classes in the reconstruction. In Table 1.1 the comparison of errors is shown for a 1D linear translation $(u_t + u_x = 0)$ of $sin(\pi x)$ computed by central scheme on overlapping cells (1.4) by use of the ENO quadratic reconstruction on two classes of cells separately

Δx	1/10	1/20	1/40	1/80	1/160	1/320
l_1 *error* E_1	0.0117	0.00147	0.000184	2.30e-05	2.88e-06	3.60e-07
order	-	2.99	3.00	3.00	3.00	3.00
l_1 *error* E_2	0.00406	0.000506	6.32e-05	7.89e-06	9.86e-07	1.23e-07
E_1/E_2	2.88	2.91	2.91	2.92	2.92	2.93

Table 1.1. E_1: reconstruction done for two classes of cells separately; E_2: reconstruction on combined overlapping cells.

(E_1) and on combined overlapping cells (E_2). Comparison of the two kinds of reconstructions are shown for the Shu-Osher problem (ShOs89) in Fig. 1.2 (c) and (d); for Lax problem in (e) and (f). Note that without characteristic decomposition, they achieve high resolution in the Shu-Osher problem while keeping a non-oscillatory profile for the Lax problem even with quite large cell size.

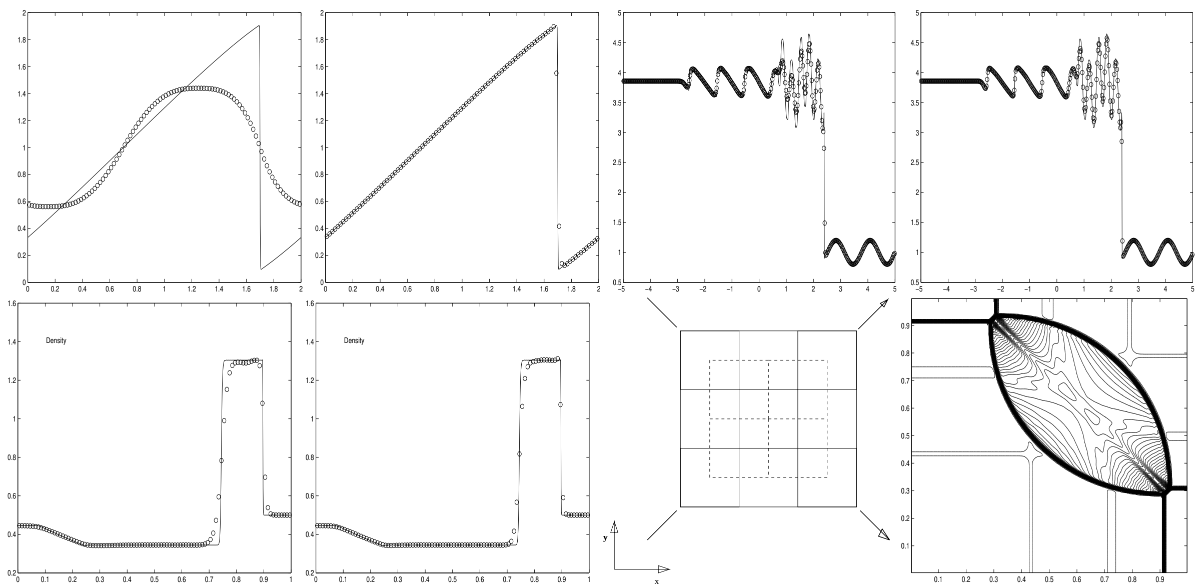

Figure 1.2. (a) Central scheme for Burgers equation without dissipation control, $\Delta t = \Delta x^2/16$, (b) with dissipation control (1.3); (c) Shu-Osher problem (ShOs89), reconstruction done for two classes of cells separately, $\Delta x = 1/40$, (d) reconstruction done for combined overlapping cells; (e)Lax problem, reconstruction done for two classes of cells separately, $\Delta x = 1/100$, (f) reconstruction done for combined overlapping cells. (g)2D overlapping cells by collapsing the staggered dual cells on two adjacent time levels to one time level; (h) 2D Riemann Problem (LaLi98) computed by DLM, $\Delta x = 1/200$. (a), (b) 2nd order; (c)-(f), (h) 3rd order. All without characteristic decomposition.

Note that in (1.3),

$$\theta\left(\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \nu^n(x)dx\right) + (1-\theta)U_i^n = U_i^n + \frac{\Delta t_n}{\Delta \tau_n}\left(\frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \nu^n(x)dx - U_i^n\right).$$

and $\Delta\tau_n = O(\Delta x)$ is due to the CFL restriction for the scheme (1.2). Therefore the local dissipative error now has a factor of Δt_n and the cumulative error will not be degenerated by choosing very small Δt_n. In the lowest order case, scheme (1.3) can be viewed as a Godunov type scheme with a Lax-Friedrich flux.

3. Central Schemes on Overlapping Cells for Convection Diffusion Equations

Consider the convection diffusion equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = \frac{\partial}{\partial x}(a(u,x,t)\frac{\partial u}{\partial x}), \quad (x,t) \in \mathcal{R} \times (0,T), \tag{1.5}$$

where $a(u,x,t) \geq 0$. Following the work of Kurganov and Tadmor (KuTa00), we can discretize equation (1.5) in the new setting as follows

$$\begin{aligned} U_i^{n+1} &= \theta(\tfrac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} \nu^n(x)dx) + (1-\theta)U_i^n - \tfrac{\Delta t_n}{\Delta x}[f(\nu^n(x_{i+1/2})) - f(\nu^n(x_{i-1/2}))] \\ &\quad + \tfrac{\Delta t_n}{\Delta x}[a(U_{i+1/2}^n, x_{i+1/2}, t_n)\tfrac{U_{i+1}^n - U_i^n}{\Delta x} - a(U_{i-1/2}^n, x_{i-1/2}, t_n)\tfrac{U_i^n - U_{i-1}^n}{\Delta x}], \\ U_{i+1/2}^{n+1} &= \theta(\tfrac{1}{\Delta x}\int_{x_i}^{x_{i+1}} \mu^n(x)dx) + (1-\theta)U_{i+1/2}^n - \tfrac{\Delta t_n}{\Delta x}[f(\mu^n(x_{i+1})) - f(\mu^n(x_i))] \\ &\quad + \tfrac{\Delta t_n}{\Delta x}[a(U_{i+1}^n, x_{i+1}, t_n)\tfrac{U_{i+3/2}^n - U_{i+1/2}^n}{\Delta x} - a(U_i^n, x_i, t_n)\tfrac{U_{i+1/2}^n - U_{i-1/2}^n}{\Delta x}], \end{aligned} \tag{1.6}$$

where $\theta = \Delta t_n/\Delta\tau_n$, $\Delta\tau_n$ is maximum time step size determined by the CFL restriction for the hyperbolic part of the equation (1.5), $\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$. We have the following stability theorem.

Theorem 2 *Let the schemes (1.2) and (1.6) start from the same time t_n with the same initial values U_i^n, $U_{i+1/2}^n$. If the scheme (1.2) is TVD from time step t_n to $t_n + \Delta\tau_n$, then the scheme (1.6) is also TVD from time t_n to $t_n + \Delta t_n$, for any $\Delta t_n \leq \frac{\Delta\tau_n \Delta x^2}{\Delta x^2 + 2a_n\Delta\tau_n}$, with*

$$a_n = \sup\{a(U_{i+1}^n, x_{i+1}, t_n), a(U_{i+1/2}^n, x_{i+1/2}, t_n) : i = 0, \pm 1, \pm 2, \cdots\}.$$

We can also obtain a semi-discrete form of (1.6) similar to (1.4). Explicit Runge-Kutta methods with larger time step size have been developed in Medovikov (Me98), Verwer (Ve96) *etc* for semi-discrete equations. Implicit-explicit Runge-Kutta time discretizations, e.g. Ascher *et. al.* (AsRuSp97), Kennedy and Carpenter (KeCa03), etc, may also be applied to the semi-discrete form of (1.6).

4. Multi Space Dimensions

For rectangular grid, dimension by dimension line methods are the most convenient high order (≥ 3) methods for multi-dimensional problems. For example, in Shu and Osher (ShOs88; ShOs89), the ENO scheme is formulated in a dimension by dimension approach; in Kurganov and Tadmor (KuTa00), the semi-discrete central scheme is also formulated in a dimension by dimension approach for multi dimensional problems. A 2D diagonal line method (DLM) is introduced in Liu (Li04) for central scheme on overlapping cells using 1D quadratic ENO reconstruction along diagonal lines on combined two classes of overlapping cells, and using dimension by dimension (diagonal) approximation to the flux derivatives (without evaluation at quadrature points). The most common (staggered) overlapping cell averages are defined as in Fig. 1.2(g). Each of the two diagonal axises passes through exactly 5 overlapping cells. We may view them as 1D overlapping cell averages as in Fig. 1.1(B) following the strategy of PPM (Colella and Woodward, CoWo84), thus the 1D quadratic ENO reconstruction can be adapted to the 1D overlapping cells. The convergence tests on 2D linear translations show that DLM has 3rd order accuracy even though the 1D quadratic ENO reconstruction along diagonal lines only has 2nd order accuracy (since the cell averages are 2D), see Table 1.2. The other 2D tests also confirm the high resolution of DLM, see Fig. 1.2(h). Note that there is no overlapping within the cells with solid boundary lines (Fig. 1.2(g)) or within the cells with dash boundary lines, therefore conventional line reconstruction can also be applied to each class of the cells separately. The reason for combining the overlapping cells in reconstruction

Δx, Δy	1/20	1/40	1/80	1/160	1/320	1/640
l_1 *error*	0.231	0.0359	0.00464	0.000583	7.30e-05	9.22e-06
order	-	2.69	2.95	2.99	3.00	2.99

Table 1.2. Convergence test for DLM for $u_t + (2u)_x + u_y = 0$, $(x, y) \in [0, 1]^2$ with periodic boundary condition. $u(x, y, 0) = sin(2\pi x + 4\pi y) + \frac{1}{3}cos(2\pi y)$, $T = 1$.

is to achieve better resolution (Li04) taking advantage of the "effectively refined grid" (in space, not time! The time step size doesn't change).

Finite volume ENO type high order reconstruction for combined overlapping cells is also important because it can be applied to unstructured meshes such as the Voronoi mesh. Second order reconstruction (e.g., MUSCL) for combined overlapping cells is straight forward as described in Li04 and has been tested to have good robustness. Higher order finite volume ENO reconstruction separately for each class of the cells (no overlapping within each class, see Fig. 1.2(g)) should be standard.

5. Central Discontinuous Galerkin Methods on Overlapping Cells for Convection Diffusion Equations

Following the general strategy of discontinuous Galerkin methods (see e.g. Lesaint and Raviart (LeRa74), Cockburn (Co98) *etc*) and the series works of Cockburn and Shu (CoSh89 *etc*), the central type discontinuous Galerkin method can be derived on overlapping cells. Consider the system of convection diffusion equations

$$\frac{\partial u_i}{\partial t} + \bigtriangledown \cdot \mathbf{f}_i(\mathbf{u}) = \bigtriangledown \cdot (A_i(\mathbf{u}, \mathbf{x}, t) \bigtriangledown u_i), \quad (\mathbf{x}, t) \in \mathcal{R}^d \times (0, T), \quad i = 1, \cdots, m, \tag{1.7}$$

where u_i is the ith component of $\mathbf{u}$, A_i is a matrix. For simplicity, assume a uniform staggered rectangular mesh (see Fig. 1.2(g) for the 2D case). The formulation for irregular staggered mesh, e.g., the Voronoi mesh which is a triangular mesh plus its dual, is similar. Let C_I, $I = (i_1, i_2, \cdots, i_d)$ be a (open) cell of a uniform rectangular mesh in R^d (the cells bounded by solid lines in Fig. 1.2(g))and $\mathbf{x}_I$ be the cell centroid. Let Δx be the cell size. Let $\mathcal{M}$ be the set of piece-wise polynomials of degree r over the cells $\{C_I\}$ with no continuity assumed across the cell boundary. Let D_J, $J = (i_1 + 1/2, i_2 + 1/2, \cdots, i_d + 1/2)$ be a (open) cell of the dual mesh which is the shift of the original mesh along the vector $(\frac{1}{2}\Delta x, \frac{1}{2}\Delta x, \cdots, \frac{1}{2}\Delta x)$ (the cells bounded by dash lines in Fig. 1.2(g)). Let $\mathbf{x}_J$ be the cell centroid of the cell D_J. Let $\mathcal{N}$ be the set of piece-wise polynomials of degree r over the cells $\{D_J\}$ with no continuity assumed across the cell boundary. The smooth solution of (1.7) will satisfy

$$\begin{aligned} \frac{d}{dt}\int_{C_I} u_i \Phi_i d\mathbf{x} = \int_{C_I} \mathbf{f}_i \cdot \bigtriangledown \Phi_i d\mathbf{x} - \int_{\partial C_I} (\mathbf{f}_i \cdot \mathbf{n}) \Phi_i ds + \\ \int_{C_I} \bigtriangledown \cdot (A_i^T \bigtriangledown \Phi_i) u_i d\mathbf{x} - \int_{\partial C_I} u_i (A_i^T \bigtriangledown \Phi_i) \cdot \mathbf{n} ds + \\ \int_{\partial C_I} \Phi_i (A_i \bigtriangledown u_i) \cdot \mathbf{n} ds, \quad \forall \Phi_i \in \mathcal{M}, \quad i = 1, \cdots, m, \ \forall I, \end{aligned} \tag{1.8}$$

$$\begin{aligned} \frac{d}{dt}\int_{D_J} u_i \Psi_i d\mathbf{x} = \int_{D_J} \mathbf{f}_i \cdot \bigtriangledown \Psi_i d\mathbf{x} - \int_{\partial D_J} (\mathbf{f}_i \cdot \mathbf{n}) \Psi_i ds + \\ \int_{D_J} \bigtriangledown \cdot (A_i^T \bigtriangledown \Psi_i) u_i d\mathbf{x} - \int_{\partial D_J} u_i (A_i^T \bigtriangledown \Psi_i) \cdot \mathbf{n} ds + \\ \int_{\partial D_J} \Psi_i (A_i \bigtriangledown u_i) \cdot \mathbf{n} ds, \quad \forall \Psi_i \in \mathcal{N}, \quad i = 1, \cdots, m, \ \forall J, \end{aligned} \tag{1.9}$$

where $\mathbf{n}$ is the unit outer normal of the corresponding cell, $\int_{\partial C_I} F(s)ds$ denotes $\int_{\partial C_I} F(s)|_{C_I} ds$. Let $U_i^n \in \mathcal{M}$ and $V_i^n \in \mathcal{N}$ both be the numerical approximations of the solution $u_i(\cdot, t_n)$. On the right hand side of the equation (1.8), replace the u_i ($\mathbf{u}$) by V_i^n ($\mathbf{V}^n$) and use the integration by part formula (similarly for (1.9)). The central discontinuous Galerkin formulation on overlapping cells with forward Euler time discretization from t_n to $t_{n+1} = t_n + \Delta t_n$ (with dissipation control as in (1.3)) is to find $U_i^{n+1} \in \mathcal{M}$ and $V_i^{n+1} \in \mathcal{N}$ such that

$$\begin{aligned}
&\int_{C_I} U_i^{n+1}\Phi_i d\mathbf{x} = \theta \int_{C_I} V_i^n \Phi_i d\mathbf{x} + (1-\theta)\int_{C_I} U_i^n \Phi_i d\mathbf{x} + \Delta t_n \{\int_{C_I} \mathbf{f}_i(\mathbf{V}^n)\cdot \bigtriangledown \Phi_i d\mathbf{x} - \\
&\quad \int_{\partial C_I} (\mathbf{f}_i(\mathbf{V}^n)\cdot \mathbf{n})\Phi_i ds + \sum_J [\int_{C_I\cap D_J} \bigtriangledown \cdot (A_i \bigtriangledown V_i^n)\Phi_i d\mathbf{x} + \\
&\quad \int_{C_I \cap \partial D_J} V_i^n (A_i^T \bigtriangledown \Phi_i)\cdot \mathbf{n} ds - \int_{C_I\cap \partial D_J} \Phi_i (A_i \bigtriangledown V_i^n)\cdot \mathbf{n} ds]\}, \\
&\qquad \forall \Phi_i \in \mathcal{M}, \;\; i = 1, \cdots, m, \; \forall I, \\
&\int_{D_J} V_i^{n+1}\Psi_i d\mathbf{x} = \theta \int_{D_J} U_i^n \Psi_i d\mathbf{x} + (1-\theta)\int_{D_J} V_i^n \Psi_i d\mathbf{x} + \Delta t_n \{\int_{D_J} \mathbf{f}_i(\mathbf{U}^n)\cdot \bigtriangledown \Psi_i d\mathbf{x} - \\
&\quad \int_{\partial D_J} (\mathbf{f}_i(\mathbf{U}^n)\cdot \mathbf{n})\Psi_i ds + \sum_I [\int_{D_J\cap C_I} \bigtriangledown \cdot (A_i \bigtriangledown U_i^n)\Psi_i d\mathbf{x} + \\
&\quad \int_{D_J \cap \partial C_I} U_i^n (A_i^T \bigtriangledown \Psi_i)\cdot \mathbf{n} ds - \int_{D_J\cap \partial C_I} \Psi_i (A_i \bigtriangledown U_i^n)\cdot \mathbf{n} ds]\}, \\
&\qquad \forall \Psi_i \in \mathcal{N}, \;\; i = 1, \cdots, m, \; \forall J,
\end{aligned} \tag{1.10}$$

where $\theta = \Delta t_n/\Delta\tau_n \le 1$, $\Delta\tau_n$ is the maximum time step size determined by the CFL restriction for the hyperbolic part of the equation (1.7) (i.e., assuming the right hand side is 0), $\Delta t_n = t_{n+1} - t_n$ is the current time step size. Note that the last two boundary integral terms of (1.8) and (1.9) are canceled out in (1.10) due to the continuity, which is different from usual discontinuous Galerkin methods. Semi-discretized version of the (1.10) can be easily obtained similar to (1.4), to which various higher order Runge-Kutta time discretization methods can be applied. If we make the corresponding diffusive fluxes in the above formula implicit (or use an implicit-explicit Runge-Kutta method to the corresponding semi-discretized version), we obtain an implicit central discontinuous Galerkin method which enables large time step size. It is the use of overlapping cells that makes it possible because otherwise the fluxes may not be able to be represented in the implicit form. Even though the two equations in (1.10) are coupled, when written down in the matrix form, one can easily eliminate one class of unknowns, say $\{U_i^{n+1}\}$, which is similar to the procedure of mixed finite element method for the incompressible Navier-Stokes equation (see. e.g. Fortin, Fo93). Therefore we are actually solving two $N \times N$ sparse systems instead of a $2N \times 2N$ sparse system.

6. Application to Elliptic Equations with Convection

Consider the equation (1.7) without the time dependence term

$$-\bigtriangledown \cdot (A_i(\mathbf{u}, \mathbf{x}) \bigtriangledown u_i) + \bigtriangledown \cdot \mathbf{f}_i(\mathbf{u}) = 0, \quad (\mathbf{x}, t) \in \mathcal{R}^d \times (0, T), \;\; i = 1, \cdots, m, \tag{1.11}$$

The corresponding central discontinuous Galerkin approach on overlapping cells can be obtained as follows: (a) write equation (1.10) in a semi-discrete form by moving the corresponding $\int_{C_I} U_i^n \Phi_i d\mathbf{x}$ or $\int_{D_J} V_i^n \Psi_i d\mathbf{x}$ terms to the left hand side and multiplying both side by $1/\Delta t_n$, then passing to the limit as $\Delta t_n \to 0$; (b) drop the time derivative term. (1.10) becomes

$$
\begin{aligned}
&\tfrac{1}{\Delta\tau}\int_{C_I}(V_i-U_i)\Phi_i d\mathbf{x}+\{\int_{C_I}\mathbf{f}_i(\mathbf{V})\cdot\bigtriangledown\Phi_i d\mathbf{x}- \\
&\quad\int_{\partial C_I}(\mathbf{f}_i(\mathbf{V})\cdot\mathbf{n})\Phi_i ds+\textstyle\sum_J[\int_{C_I\cap D_J}\bigtriangledown\cdot(A_i\bigtriangledown V_i)\Phi_i d\mathbf{x}+ \\
&\quad\int_{C_I\cap\partial D_J}V_i(A_i^T\bigtriangledown\Phi_i)\cdot\mathbf{n}ds-\int_{C_I\cap\partial D_J}\Phi_i(A_i\bigtriangledown V_i)\cdot\mathbf{n}ds]\}=0, \\
&\quad\forall\Phi_i\in\mathcal{M},\;\; i=1,\cdots,m,\;\forall I, \\
&\tfrac{1}{\Delta\tau}\int_{D_J}(U_i-V_i)\Psi_i d\mathbf{x}+\{\int_{D_J}\mathbf{f}_i(\mathbf{U})\cdot\bigtriangledown\Psi_i d\mathbf{x}- \\
&\quad\int_{\partial D_J}(\mathbf{f}_i(\mathbf{U})\cdot\mathbf{n})\Psi_i ds+\textstyle\sum_I[\int_{D_J\cap C_I}\bigtriangledown\cdot(A_i\bigtriangledown U_i)\Psi_i d\mathbf{x}+ \\
&\quad\int_{D_J\cap\partial C_I}U_i(A_i^T\bigtriangledown\Psi_i)\cdot\mathbf{n}ds-\int_{D_J\cap\partial C_I}\Psi_i(A_i\bigtriangledown U_i)\cdot\mathbf{n}ds]\}=0, \\
&\quad\forall\Psi_i\in\mathcal{N},\;\; i=1,\cdots,m,\;\forall J.
\end{aligned}
\tag{1.12}
$$

Here $\Delta\tau$ is inherited from $\Delta\tau_n$ in (1.10) and can be determined similarly. This central discontinuous Galerkin formulation is locally conservative. Existing locally conservative finite element methods to this type of equation are control volume methods (see Baliga and Patankar (BaPa80), Liu and McCormick (LiMc88), Cai and McCormick (CaMc90), *etc*), skew-symmetric discontinuous Galerkin methods (see e.g. Oden *et. al.*, OdBaBa98, Baumann and Oden, BaOd99, Riviere *et. al.*, RiWhGi01) and mixed methods (see e.g. BrFo91). The distinctive feature of the central discontinuous Galerkin formulation on overlapping cells would be its central type treatment to the convection without using any upwind information. The use of overlapping cells allows a convex combination of the overlapping cell elements to control the dissipation from staggering and the resulting semi-discrete form of (1.10) makes (1.12) possible. A benefit is that this formulation doesn't require the diffusive matrix A to be symmetric.

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