Symmetric Error Estimates for Moving Mesh Galerkin Methods for Advection-Diffusion Equations *

Todd F. Dupont † Yingjie Liu ‡

Abstract

This work tries to increase our understanding of why moving mesh methods often work very well. It combines techniques from the symmetric error estimates (SEEs) of Dupont [4] and Bank and Santos [1] with ideas that motivated the analysis of a modified method of characteristics by Douglas and Russell [2]. By changing the usual time derivative to a time derivative along approximate characteristics in the SEE norm, the symmetric error estimate in [1] can be improved. In addition, by introducing yet another SEE norm which is more strongly mesh-dependent we provide another SEE which provides different insights into the convergence of these methods; one symmetric error estimate that is presented can be used to derive optimal order L^2 convergence in certain settings.

Keywords. Galerkin methods, parabolic equations, finite element, moving mesh

AMS subject classification. 65M60, 65M12

---

*Supported by the ASCI Flash Center at the University of Chicago under DOE contract B341495 and by the MRSEC Program of the National Science Foundation under Award Number DMR-9808595.

†t-dupont@uchicago.edu, tel. 773-702-3485, fax 773-702-8487, Department of Computer Science, The University of Chicago, Chicago, Illinois 60637, USA

‡yingjie@ams.sunysb.edu, tel. 631-632-8552, Dept of Applied Math, SUNY at Stony Brook, NY 11794, USA

1 Introduction

Moving mesh finite element methods have been known for quite a while [5, 6] and they are increasingly used in practice, but the analytical understanding of these methods is far from complete.

A symmetric error estimate is, roughly speaking, a statement of the form, if the error *can be* small in a certain norm then it *is* small in that same norm. Somewhat more precisely there is a norm, $\|\cdot\|$, and a constant, C, such that

$$\|error\| \leq C\|\text{best approximation error}\|,$$

where the left-hand side measures the error in the method at hand and the right-hand side reflects the distance between the true solution and the function spaces used in the method. Of course, we need control on C if such an estimate is to be informative.

The results in section 2 of [4], section 3 of [1], and section 3 of this work give bounds of this type. There are two things that distinguish the bounds given here from the earlier work. The first is that here the constant, C, does not increase as the advective term increases in size, provided the mesh movement approximates the advective term well in a sense that is made precise. Hence these results make it more clear that the mesh movement is actually modeling the advection. The second is that the norm in section 3 involves the convective derivative instead of the partial with respect to time, and as Douglas and Russell pointed out in [2] for advection dominated problems the convective derivative will be much smoother, and therefore easier to approximate well. To give credit where it is due, Bank and Santos noted in [1] that in part of their analysis the constants could be made independent of the size of the advective term, and they also noted the similarity of the difference equations to the modified method of characteristics [2].

While symmetric error estimates for parabolic equations have a certain attractiveness in the simplicity of the statement that they make, it is sometimes hard to see the precise meaning of the result. In the case of Galerkin methods for elliptic equations one has a symmetric error estimate in the H^1-norm, a statement that is relatively easy to understand. In the case of parabolic equations, symmetric error estimates [3, 4, 1] involve combining several norms and semi-norms, in the case of [4] for example the $\|\cdot\|$ is made up from two norms and a semi-norm: the maximum in time of the L^2-norm in space, the L^2 in time norm of the H^1-norm in space, and the L^2 in time norm of the discrete H^{-1} semi-norm in space of the time derivative. In one of the analogous results here, the H^1-norm is replaced by the "discrete H^1"-norm, i.e., the H^1-norm of the H^1 projection into the space. It might appear at first that weakening the norms is not an advantage, but it actually highlights the importance of the only remaining norm to such a degree that one can get optimal order L^2 convergence in some contexts. In a sense the SEE that comes from this norm is a way to combine the techniques of [4] with those of Wheeler [7]. We view this as one of the most interesting results of this work.

In section 2 we give the advection-diffusion problem whose approximate solution we are studying here, and we define a continuous-time moving mesh method in terms of a "convected time derivative". In section 3 we give three symmetric error bounds for the continuous-time case. Then we present a symmetric error estimate for a discrete time case. In sections 4 and 5 we give two optimal order L^2 error bounds that follow from the results of section 3.

2 Model Problem and a Moving Mesh Galerkin Method

Consider the following advection-diffusion model problems on $Q = \Omega \times (0,T)$,

$$\begin{cases} \partial_t u - \bigtriangledown \cdot (a \bigtriangledown u) + v \cdot \bigtriangledown u + cu = f, & on\ Q, \\ \frac{\partial u}{\partial \nu} = g, & on\ \Gamma_N \times (0,T), \\ u = 0, & on \Gamma_D \times (0,T), \\ u = u_0, & for\ t = 0, \end{cases} \tag{1}$$

where $a(x,t), v(x,t), c(x,t), f(x,t)$, and $g(x,t)$ are smooth and bounded and $0 < a_0 \le a \le a_1$ for some constants $a_0, a_1 > 0$. Ω is a bounded domain in R^d. For simplicity, we assume that Ω is a fixed polyhedron. Γ_D, Γ_N are parts of the boundary $\partial\Omega$ such that $\Gamma_D \cap \Gamma_N = \emptyset$, $\Gamma_D \cup \Gamma_N = \partial\Omega$ and Γ_D is closed. Suppose that $\bar{D} = \cup D_i$ is a fixed polyhedron where the D_i's are closed sets with nonvoid interior such that the interiors of the D_i's are disjoint. We need few restrictions on the D_i's for much of the argument, but to keep the discussion simple we suppose that each D_i is a simplex and that they form a tessellation of D. We suppose that there is a continuous mapping $\mathcal{G}$ from $\bar{D} \times [0,T]$ onto $\bar{\Omega}$ such that: (1) for each t, $\mathcal{G}(\cdot,t)$ is a $1-1$ piecewise linear mapping (with respect to $\{D_j\}$) of $\bar{D}$ onto $\bar{\Omega}$; and (2) $\mathcal{G}$ is continuously

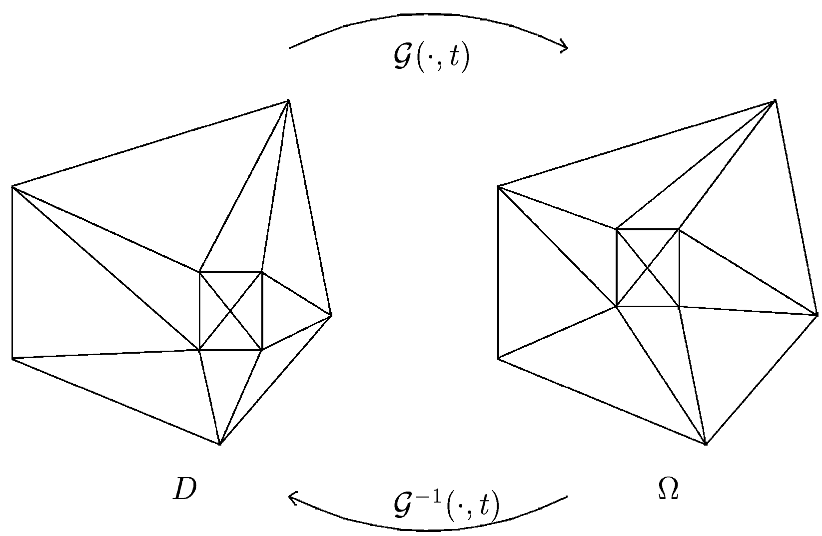

Figure 1: Moving mesh as a time dependent mapping $\mathcal{G}$.

differentiable on each $D_i \times [0,T]$. We also suppose that $\partial D = \gamma_D \cup \gamma_N$ and that $\Gamma_D = \mathcal{G}(\gamma_D, t)$ and $\Gamma_N = \mathcal{G}(\gamma_N, t)$. We denote by $\Omega_i = \Omega_i(t)$ the image of D_i under $\mathcal{G}(\cdot,t)$. Let $\mathcal{M}_D$ be a finite dimensional subspace of $H^1(D)$ so that each function in $\mathcal{M}_D$ vanishes on γ_D; then the finite element space on Ω is defined by $\mathcal{M}(t) = \{\phi(x,t) : \phi(\mathcal{G}(\cdot,t),t) \in \mathcal{M}_D\}$. It is sometimes convenient to think of this moving mesh as being generated by a mapping of Ω onto itself. (See Fig. 1.)

Let $\mathcal{G}^{-1} = \mathcal{G}^{-1}(\cdot,t)$ denote the inverse of $\mathcal{G}$ as a map of $\bar{D}$ onto $\bar{\Omega}$ with t fixed; so this function can be thought of as being defined on $\bar{Q}$. Let $\mathcal{G}_t$ be the partial derivative of $\mathcal{G}$ with respect to t. The finite element mesh is advected with a flow that is given by

$$\dot{x}(t) = \mathcal{G}_t(\mathcal{G}^{-1}(x,t),t).$$

Denote a particular directional derivative as follows,

$$\frac{D}{Dt}F(x,t) = \frac{\partial}{\partial t}F(x,t) + w \cdot \nabla_x F(x,t).$$

where $w(x,t)$ is a differentiable vector function such that $w \cdot \nu = 0$ on Γ_N for $t \geq 0$, ν is the unit outer normal of $\partial\Omega$.

We will use $\|\cdot\|_k$ as the norm on the Sobolev space $H^k(\Omega)$; for domains R other than Ω we will use the more explicit notation $\|\cdot\|_{H^k(R)}$. The norm and inner product on $L^2(\Omega)$ will be denoted as $\|\cdot\|$ and $(\cdot,\cdot)$, respectively.

The exact solution of (1) will satisfy

$$(\frac{Du}{Dt},\psi)+(a\bigtriangledown u,\bigtriangledown\psi)+((v-w)\cdot\bigtriangledown u,\psi)+(cu,\psi)=(f,\psi)+\int_{\Gamma_N} g\psi ds, \quad (2)$$

for any $\psi \in H^1(\Omega)$. We are looking for $U \in \mathcal{M}(t)$ such that

$$(\frac{DU}{Dt},\phi)+(a\bigtriangledown U,\bigtriangledown\phi)+((v-w)\cdot\bigtriangledown U,\phi)+(cU,\phi)=(f,\phi)+\int_{\Gamma_N} g\phi ds, \quad (3)$$

for any $\phi \in \mathcal{M}(t)$. The inclusion of the convective derivative here is not really a change from the method discussed in [4], we have just added and subtracted a term. However, it reflects a change in the way that we think about and analyze the method. We will take the initial value for U to be the L^2-projection of u_0 into $\mathcal{M}(0)$.

3 Symmetric Error Bounds

First we get a basic relation that will be used in bounding the error. Taking $\Psi \in \mathcal{M}(t)$ and setting $\Phi = U - \Psi \in \mathcal{M}(t)$ and $\eta = u - \Psi$, gives for $\phi \in \mathcal{M}(t)$

$$\begin{aligned} &(\frac{D\Phi}{Dt},\phi)+(a\bigtriangledown\Phi,\bigtriangledown\phi)+((v-w)\cdot\bigtriangledown\Phi,\phi)+(c\Phi,\phi) \\ &\quad =(\tfrac{D\eta}{Dt},\phi)+(a\bigtriangledown\eta,\bigtriangledown\phi)+((v-w)\cdot\bigtriangledown\eta,\phi)+(c\eta,\phi). \end{aligned} \quad (4)$$

From the definition of directional derivative we have the following equality which we use in the energy-type arguments used later.

Lemma 1 *Suppose that $\phi(t) \in \mathcal{M}(t)$, and that ϕ is differentiable with respect to t as a map into $L^2(\Omega)$. Then*

$$(\frac{D\phi}{Dt},\phi)=\frac{1}{2}\{\frac{d}{dt}\|\phi\|^2-\int_\Omega \phi^2 \bigtriangledown_x \cdot w dx\}.$$

Proof:

$$\begin{array}{rcl} \frac{d}{dt}\|\phi\|^2 & = & 2\int_\Omega \phi_t\phi dx \\ & = & 2\int_\Omega \frac{D\phi}{Dt}\phi dx - 2\int_\Omega (w\cdot\bigtriangledown\phi)\phi dx \\ & = & 2\int_\Omega \frac{D\phi}{Dt}\phi dx + \int_\Omega \phi^2(\bigtriangledown\cdot w)dx - \int_{\Gamma_N}\phi^2 w\cdot\nu ds \\ & = & 2\int_\Omega \frac{D\phi}{Dt}\phi dx + \int_\Omega \phi^2(\bigtriangledown\cdot w)dx. \end{array} \tag{5}$$

□

Define the mesh-dependent semi-norm $\|\cdot\|_{(-1,\mathcal{M}(t))}$ by

$$\|u\|_{(-1,\mathcal{M}(t))} = \sup_{\phi\in\mathcal{M}(t),\phi\neq 0}\frac{|(u,\phi)|}{\|\phi\|_1}.$$

For X a normed space and v a function that maps $(0,T)$ into X, let

$$\|v\|_{L^p(0,T;X)}$$

denote the L^p-norm on the interval $(0,T)$ of the X-norm of v. The first symmetric error estimate will be given norm $|\!|\!|\cdot|\!|\!|$ defined by

$$|\!|\!|v|\!|\!|^2 = \|v\|^2_{L^\infty(0,T;L^2(\Omega))} + \|v\|^2_{L^2(0,T;H^1(\Omega))} + \int_0^T \|\frac{Dv}{Dt}\|^2_{(-1,\mathcal{M}(t))}dt.$$

Theorem 1 *Suppose that there eixist constant c_1 and c_2 such that for all $(x,t)\in Q$*

$$\bigtriangledown_x\cdot w(x,t)\leq c_1, \qquad |w-v|(x,t)\leq c_2. \tag{6}$$

Then there is a constant C depending only on c_1, c_2, T and bounds on the coefficients a and c such that, for any smooth function Ψ from $[0,T]$ into $L^2(\Omega)$ with $\Psi(t)\in\mathcal{M}(t)$,

$$|\!|\!|u-U|\!|\!| \leq C|\!|\!|u-\Psi|\!|\!|.$$

Proof: By using $\phi=\Phi$ in equation (4) we see then that

$$\frac{d}{dt}\|\Phi\|^2 + a_0\|\Phi\|_1^2 \leq C\{\|\Phi\|^2 + \|\frac{D\eta}{Dt}\|^2_{(-1,\mathcal{M}(t))} + \|\eta\|_1^2\}. \tag{7}$$

This estimate and Gronwall's inequality give that

$$\|\Phi\|^2_{L^\infty(0,T;L^2(\Omega))} + a_0\|\Phi\|^2_{L^2(0,T;H^1(\Omega))} \leq C\left(\|\Phi(0)\|^2_{L^2(\Omega)} + |\!|\!|\eta|\!|\!|^2\right). \tag{8}$$

Also, for any $\phi \in \mathcal{M}(t)$, (4) gives that

$$(\frac{D\Phi}{Dt}, \phi) \leq C\{\|\Phi\|_1 + \|\frac{D\eta}{Dt}\|_{(-1,\mathcal{M}(t))} + \|\eta\|_1\}\|\phi\|_1. \tag{9}$$

Therefore

$$\int_0^T \|\frac{D\Phi}{Dt}\|^2_{(-1,\mathcal{M}(t))} dt \leq C|\!|\!|\eta|\!|\!|^2. \tag{10}$$

Since $U(0)$ is the L^2 projection into $\mathcal{M}(0)$ of u_0 we see that $\|\Phi(0)\| \leq \|\eta(0)\|$. Hence $|\!|\!|\Phi|\!|\!| \leq C|\!|\!|\eta|\!|\!|$. The triangle inequality then gives that $|\!|\!|u - U|\!|\!| \leq C|\!|\!|u - \Psi|\!|\!|$. $\square$

In the application of Gronwall's inequality one gets exponential growth in time of the estimate of the error, unless there is sufficient dissipation in the equation to counter it. If we let c_0 be a bound for the absolute value of $c(x,t)$ on Q, then the arithmetic of the proof gives that the constant C of (8) contains a factor $exp(KT)$, where K can be of the form

$$K = 3c_0 + c_1 + c_2 + a_0/3 + 3c_2^2/a_0.$$

Hence, if c_2 is large and a_0 is small, this constant is very big. An interesting side light of the above calculation is that most of it is local, so that the important quantity for most parts of the estimate is the maximum of $|v - w|^2/a$. This would lead one to conjecture that in parts of the problem where diffusion is small the directional derivatives that we bring into the estimation should be very close to the ones that point in characteristic directions.

The function w in the definition of the directional derivative should be chosen so that $|\!|\!|u - \Psi|\!|\!|$ in the above theorem is small. To illustrate how this might be done we consider the case in which $\mathcal{M}(t)$ is the space of continuous piecewise linear functions over a triangular mesh given by the Ω_i's. If we take $w = \dot{x}$, then nodal interpolation commutes with the convective differentiation; i.e., $\frac{DIu}{Dt} = I\frac{Du}{Dt}$, where Iu is the nodal interpolant of u. Therefore

$$\|\frac{D(u - Iu)}{Dt}\|_{(-1,\mathcal{M}(t))} \leq \|\frac{D(u - Iu)}{Dt}\| \leq C\left(\sum_i h_i^4 \|\frac{Du}{Dt}\|^2_{H^2(\Omega_i)}\right)^{\frac{1}{2}},$$

where h_i is the diameter of Ω_i. Here we emphasize that the norm involved is applied to the convective derivative, which can be a much smoother function than the usual partial time derivative.

Next we weaken the norm used in the previous theorem in two different ways to get somewhat different results.

Let $c_3 = (a_0 + c_2/a_0)/2$. Set

$$\mathcal{B}(\varphi, \psi) = (a \bigtriangledown (\varphi), \bigtriangledown\psi) + ((v - w) \cdot \bigtriangledown\varphi, \psi) + c_3(\varphi, \psi).$$

It is easy to check that for any $\varphi \in H^1(\Omega)$

$$\mathcal{B}(\varphi, \varphi) \geq \frac{a_0}{2}\|\varphi\|_1^2. \tag{11}$$

We define a linear projection $P_1 : \ H^1(\Omega) \to \mathcal{M}(t)$ by

$$\mathcal{B}(v - P_1 v, \phi) = 0, \tag{12}$$

for all $\phi \in \mathcal{M}(t)$ Now we can define a new norm $|\!|\!| \cdot |\!|\!|_0$ in which the H^1 part of the previous norm has been weakened to be a semi-norm:

$$|\!|\!| v |\!|\!|_0^2 = \|v\|^2_{L^\infty(0,T;L^2(\Omega))} + \|P_1 v\|^2_{L^2(0,T;H^1(\Omega))} + \int_0^T \|\frac{Dv}{Dt}\|^2_{(-1,\mathcal{M}(t))} dt. \tag{13}$$

The mnemonic for the use of the subscript 0 is that his norm emphasizes the H^0 or L^2 part of the norm.

Another norm can also be defined to put more weight on the $L^2(H^1)$ part of $|\!|\!| \cdot |\!|\!|$ by weakening the $L^\infty(L^2)$ part. Let P_0 be the L^2 projection onto $\mathcal{M}(t)$. Set

$$|\!|\!| v |\!|\!|_1^2 = \|P_0 v\|^2_{L^\infty(0,T;L^2(\Omega))} + \|v\|^2_{L^2(0,T;H^1(\Omega))} + \int_0^T \|\frac{Dv}{Dt}\|^2_{(-1,\mathcal{M}(t))} dt. \tag{14}$$

Theorem 2 *If the conditions of Theorem 1 hold, then there is a constant $C \geq 0$ depending only on c_1, c_2, T and bounds on the coefficients a, c such that for any smooth function Ψ from $[0, T]$ into $L^2(\Omega)$ with $\Psi(t) \in \mathcal{M}(t)$,*

$$\begin{aligned} |\!|\!| u - U |\!|\!|_0 &\leq C |\!|\!| u - \Psi |\!|\!|_0, \\ |\!|\!| u - U |\!|\!|_1 &\leq C |\!|\!| u - \Psi |\!|\!|_1. \end{aligned}$$

Proof: Because the test function ϕ in (4) is in the space $\mathcal{M}(t)$ we can rewrite that relation as

$$\begin{aligned} &(\tfrac{D\Phi}{Dt}, \phi) + (a \bigtriangledown \Phi, \bigtriangledown\phi) + ((v - w) \cdot \bigtriangledown\Phi, \phi) + (c\Phi, \phi) \\ &= (\tfrac{D\eta}{Dt}, \phi) + \mathcal{B}(P_1\eta, \phi) + ((c - c_3)\eta, \phi). \end{aligned} \tag{15}$$

This gives the following analog of (7):

$$\frac{d}{dt}\|\Phi\|^2 + a_0\|\Phi\|_1^2 \le C\{\|\Phi\|^2 + \|\frac{D\eta}{Dt}\|^2_{(-1,\mathcal{M}(t))} + \|P_1\eta\|_1^2 + \|\eta\|^2\}. \quad (16)$$

Since $P_1\Phi = \Phi$, this becomes

$$\frac{d}{dt}\|\Phi\|^2 + a_0\|P_1\Phi\|_1^2 \le C\{\|\Phi\|^2 + \|\frac{D\eta}{Dt}\|^2_{(-1,\mathcal{M}(t))} + \|P_1\eta\|_1^2 + \|\eta\|^2\}. \quad (17)$$

The estimate (9) becomes

$$(\frac{D\Phi}{Dt}, \phi) \le C\{\|P_1\Phi\|_1 + \|\frac{D\eta}{Dt}\|_{(-1,\mathcal{M}(t))} + \|P_1\eta\|_1 + \|\eta\|\}\|\phi\|_1. \quad (18)$$

The relations (17) and (18) give the bound for the $\|\!|\cdot|\!\|_0$ norm, just as in the proof of Theorem 1.

Examination of (7) shows that the η term in (8) can be replaced by $|\!|\!|\eta|\!|\!|_1$. The fact that $P_0\Phi = \Phi$ gives

$$\|P_0\Phi\|^2_{L^\infty(0,T;L^2(\Omega))} + a_0\|\Phi\|^2_{L^2(0,T;H^1(\Omega))} \le C\left(\|P_0\Phi(0)\|^2_{L^2(\Omega))} + |\!|\!|\eta|\!|\!|_1^2\right).$$

Next from (9) and the above relation we see that the analog of (10) holds with $|\!|\!|\eta|\!|\!|$ replaced by $|\!|\!|\eta|\!|\!|_1$. Also, the use of $U(0) = P_0u(0)$ gives that $\|\Psi(0)\| = \|\eta(0)\|$. Combining these observations completes the proof of the second inequality in the theorem. □

Next we examine a fully discrete scheme. In this case we restrict ourselves to the case $w(x,t) = \dot{x}$. Following [1], for a given partition $P = \{t_0 = 0, t_1, \cdots, t_{n-1}, t_n = T\}$ of $[0,T]$, consider $\mathcal{G}(s,t)$ to be linear in t for $t \in [t_{i-1}, t_i)$, for any i, and continuous in t on the whole of $[0,T]$. Let M be the collection of functions $\phi(x,t)$ on Q such that $\phi(\cdot,t) \in \mathcal{M}(t)$ for any $t \in [0,T]$, piecewise continuous in t and linear along the trajectory of mesh movement, i.e.

$$\phi(\mathcal{G}(s,t),t) = \phi(\mathcal{G}(s,t_{j-1}),t_{j-1}) + \theta[\phi(\mathcal{G}(s,t_j),t_j) - \phi(\mathcal{G}(s,t_{j-1}),t_{j-1})],$$

where $t = t_{j-1} + \theta(t_j - t_{j-1})$, for any $\theta \in [0,1]$. The following relation holds for any $t \in (t_{i-1}, t_i)$, $s \in D_j$, for all i, j:

$$\frac{\phi(\mathcal{G}(s,t_i),t_i) - \phi(\mathcal{G}(s,t_{i-1}),t_{i-1})}{t_i - t_{i-1}} = \frac{D\phi}{Dt}(\mathcal{G}(s,t),t).$$

Note that $\frac{D\phi}{Dt}$ is just the same as in the continuous time case on each (t_{i-1}, t_i) with the restriction that $w = \dot{x}$, but it also has a discrete form in this special case. It is clear that functions in M are defined by their values at the t_j's, so to define the approximate solution we need only say how it is computed at the times t_j.

The time discrete approximate solution $U \in M$ is such that $U(0)$ is the L^2 projection of $u(0)$ onto $\mathcal{M}(0)$ and, for $t = t_j-$,

$$(\frac{DU}{Dt}, \phi) + (a \bigtriangledown U, \bigtriangledown \phi) + ((v-w) \cdot \bigtriangledown U, \phi) + (cU, \phi) = (f, \phi) + \int_{\Gamma_N} g\phi ds, \quad (19)$$

for any $\phi \in \mathcal{M}(t_j)$, $j = 1, 2, \cdots, n$. Let

$$|\!|\!|v|\!|\!|_{0,d}^2 = \max_{0 \le j \le n} \|v(t_j)\|^2 + \sum_{j=1}^{n} (t_j - t_{j-1}) \{ \|P_1 v(t_j)\|_1^2 + \| \frac{D}{Dt} v(t_j-)\|_{(-1,\mathcal{M}(t_j))}^2 \},$$

we have the following theorem parallel to Theorem 3.1 in [1].

Theorem 3 *Let $\mathcal{D}(t_j)$ denote the piecewise constant function $|det(\bigtriangledown_s \mathcal{G})|$ on D. If there are constants $c_1, c_2 > 0$ independent of the mesh so that*

$$\frac{\mathcal{D}(t_j) - \mathcal{D}(t_{j-1})}{t_j - t_{j-1}} \le c_1 \mathcal{D}(t_{j-1}),$$

for $1 \le j \le n$, and $|w - v|(x, t) \le c_2$ for all $(x, t) \in Q$, then there is a constant $C \ge 0$ depending only on c_1, c_2, T and bounds of coefficients a, c such that $|\!|\!|u - U|\!|\!|_{0,d} \le C |\!|\!|u - \Psi|\!|\!|_{0,d}$ for any $\Psi \in M$.

Proof: The fully discretized scheme yields an error similar to (15), with $t = t_j-$. Let $\phi = \Phi(t_j)$ in the analog of (15), and use an argument like that in [1] to get

$$\begin{array}{rl}(\frac{D\Phi(t_j-)}{Dt}, \Phi(t_j)) \ge & \frac{1}{2\Delta t_j}(\|\Phi(t_j)\|^2 - \|\Phi(t_{j-1})\|^2) \\ & -\frac{c_1}{2}\|\Phi(t_{j-1})\|^2,\end{array} \quad (20)$$

where $\Delta t_j = t_j - t_{j-1}$. We then have

$$\begin{array}{l}\frac{1}{\Delta t_j}(\|\Phi(t_j)\|^2 - \|\Phi(t_{j-1})\|^2) + \frac{1}{2}a_0 \| \bigtriangledown \Phi(t_j)\|^2 \\ \le C\{\|\frac{D\eta(t_j-)}{Dt}\|_{(-1,\mathcal{M}(t_j))}^2 + \|P_1\eta(t_j)\|_1^2 + \|\Phi(t_j)\|^2 + \|\eta(t_j)\|^2\}.\end{array} \quad (21)$$

From the discrete Gronwall's inequality one obtains the following:

$$\|\Phi(t_j)\|^2 + \frac{1}{2}a_0 \sum_{i=1}^{j} \Delta t_i \| \bigtriangledown \Phi(t_i)\|^2 \le C\|\eta\|_{0,d}^2. \tag{22}$$

Also from the analog to (15) at $t = t_j-$, for any $\phi \in \mathcal{M}(t_j)$,

$$\begin{aligned}(\tfrac{D\Phi(t_j-)}{Dt}, \phi(t_j)) \quad \le \quad & C\{\| \bigtriangledown \Phi(t_j)\| + \|\Phi(t_j)\| + \|\tfrac{D\eta(t_j-)}{Dt}\|_{(-1,\mathcal{M}(t_j))} + \\ & \|P_1\eta(t_j)\|_1 + \|\eta(t_j)\|\}\|\phi(t_j)\|_1.\end{aligned} \tag{23}$$

With the help of (22), (23) becomes

$$\sum_{i=1}^{j} \Delta t_i \|\frac{D\Phi(t_i-)}{Dt}\|_{(-1,\mathcal{M}(t_i))} \le C\|\eta\|_{0,d}. \tag{24}$$

Finally combine (22), (24), and a triangle inequality to complete the proof. □

4 An Optimal Order L^2 Error Estimate

In this section, we prove the following optimal order error estimate for the one-space dimensional, continuous time case. We will take $\mathcal{M}_D$ to be the space of continuous piecewise linear functions over a mesh $0 = s_0 < s_1 < \ldots < s_m = 1$ on the reference domain $D = [0,1]$, so $\mathcal{M}(t)$ is just the space of continuous functions which are polynomials of degree at most one on each interval $\Omega_i = [x_{i-1}, x_i]$, with $x_i(t) = \mathcal{G}(s_i, t)$. Take $w = \dot{x}$. Let h_i denote the length of Ω_i, and note that $\dot{x}$ is a continuous piecewise linear function over the mesh. The following theorem gives an optimal order error estimate in which the error bound depends on the bounds of the difference between the growth rate of the length of each element with respect to time and the rate of "compression" of the exact solution, i.e. c_1; the difference between the convection velocity and the velocity of mesh movement, i.e. c_2; and other bounds of the coefficients of (1). Most importantly, the error bound does not depend on the convection velocity v, which shows an advantage of mesh movement.

Theorem 4 *If there are constants $c_1, c_2, c_3 > 0$ so that $\|\partial_x(v-\dot{x})\|_\infty \le c_1$, $\|v-\dot{x}\|_\infty \le c_2$ and $\max_i \|\partial_x a\|_{L^\infty(\Omega_i)} \le c_3$ for all $t \in [0,T]$, then there is a constant $C(c_1, c_2, c_3, a_0, a_1, c, T; \Omega)$ such that*

$$\begin{aligned}\|u-U\|(t) \le \quad & C\{\|(\textstyle\sum_i h_i^4 \|u\|^2_{H^2(\Omega_i)})^{1/2}\|_{L^\infty[0,T]} \\ & +\|(\textstyle\sum_i h_i^4 \|\frac{Du}{Dt}\|^2_{H^2(\Omega_i)})^{1/2}\|_{L^2[0,T]}\}\end{aligned} \tag{25}$$

for any $0 \le t \le T$.

Proof: The proof is an application of Theorem 2 using $|\!|\!| \cdot |\!|\!|_0$. Since $|\!|\!| u-U |\!|\!|_0$ dominates the term we want to bound, it suffices to show that $|\!|\!| u-\Psi |\!|\!|_0$ can be bounded by terms on the right-hand side of (25). We choose Ψ to be the nodal interpolant Iu of u. The estimate of $\|u-\Psi\|_{L^\infty(0,T;L^2(\Omega))}$ is straightforward. The observation that $\frac{D}{Dt}$ commutes with interpolation means that $\|\frac{D}{Dt}(u-\Psi)\|_{L^2(0,T;L^2(\Omega))}$ can be bounded by the terms on the right-hand side of (25); hence the weaker semi-norm on $\frac{D}{Dt}(u-\Psi)$ is also bounded.
The $H^1(\Omega)$-norm of $P_1(u-\Psi)$ can be bounded as follows. For any $\phi \in \mathcal{M}(t)$,

$$\begin{aligned} & \mathcal{B}(P_1(u-\Psi),\phi) = \mathcal{B}(u-\Psi,\phi) \\ = \quad & \textstyle\sum_i \int_{\Omega_i} a\partial_x(u-\Psi)\, \partial_x\phi dx + \sum_i \int_{\Omega_i}(v-\dot{x})\partial_x(u-\Psi)\, \phi dx \\ & +c_3(u-\Psi,\phi) \\ = \quad & -\textstyle\sum_i \int_{\Omega_i}(u-\Psi)\partial_x a\, \partial_x\phi dx - \sum_i \int_{\Omega_i}(u-\Psi)\{\phi\partial_x(v-\dot{x}) \\ & +(v-\dot{x})\partial_x\phi\}dx + c_3(u-\Psi,\phi) \\ \le \quad & \|\partial_x a\|_{L^\infty(\Omega)}\|u-\Psi\|\|\phi\|_1 + \|\partial_x(v-\dot{x})\|_{L^\infty(\Omega)}\|u-\Psi\|\|\phi\| \\ & +\|v-\dot{x}\|_{L^\infty(\Omega)}\|u-\Psi\|\|\phi\|_1. \end{aligned} \tag{26}$$

Using the coercivity of $\mathcal{B}(\cdot,\cdot)$ (see (11)) and taking $\phi = P_1(u-\Psi)$ we get that

$$\|P_1(u-\Psi)\|_1 \le C\|u-\Psi\|. \qquad \square$$

Note that the integration by parts was done subinterval by subinterval so a needs only to be locally smooth. The approximation results in this section are more local than we can prove in the general case studied in the next section.

5 Optimal Order $L^2(\Omega)$ Error Estimate for General Space Dimension

In this section we return to the d-dimensional case. There will be several situations in which we need to use surface integrals on the elements Ω_i; we will use 2-dimensional terminology and refer to these as integrals over the edges. Thus an edge is the intersection of $\bar{\Omega}_i$'s with positive $(d-1)$-dimensional measure. Consider the Dirichlet problem, $\Gamma_N = \emptyset$, and take $w = \dot{x}$. Denote by e_j the edge between two adjacent elements and by n_{e_j} a normal to e_j, and define the jump operator $[\cdot]$ across the edge e_j by

$$[\mathcal{F}](x) = \lim_{\epsilon \to 0+} \{\mathcal{F}(x + \epsilon n_{e_j}) - \mathcal{F}(x - \epsilon n_{e_j})\}, \forall x \in e_j.$$

Assume that Ω and a are such that the Diriclet problem has uniform H^2 regularity; i.e., there is a constant C such that for any $t \in [0,T]$, $q \in L^2(\Omega)$, there exists a $\xi \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfying

$$\int_\Omega a \bigtriangledown \xi \cdot \bigtriangledown \eta dx = \int_\Omega q \eta dx, \ \forall \eta \in H_0^1(\Omega), \tag{27}$$

and $\|\xi\|_2 \le C\|q\|$.

Suppose that $\mathcal{M}_D$ consists of a space of continuous piecewise polynomials of degree at most r. We assume that there is a constant $\tilde{C}$ such that for any $t \in [0,T]$ and for any $\xi \in H_0^1(\Omega) \cap \{\Pi_i H^s(\Omega_i)\}$, $s \ge 2$,

$$\inf_{\phi \in \mathcal{M}(t)} \|\xi - \phi\|_l^2 \le \tilde{C} \sum_i h_i^{2(\min\{r+1,s\}-l)} \|\xi\|_{H^s(\Omega_i)}^2, \ l = 0, 1,$$

where h_i is the diameter of the element Ω_i. Let h denote $\max_i h_i$.

In this section we need bounds on $\bigtriangledown \dot{x}$, the Jacobian of the function $\dot{x}$ with respect to x. We will use the norm on matrices that is induced by the Euclidean norm on vectors. In particular $\| \bigtriangledown \dot{x}\|_\infty$ is the $L^\infty(\Omega)$ norm of the norm of the matrix $\bigtriangledown \dot{x}$.

We have the following optimal order estimate for the $L^2(\Omega)$-norm of the error. While it looks like a generalization of Theorem 4 to higher dimensional spaces, there are differences. The hypotheses are stronger here, and the result is not quite so local.

Theorem 5 *Suppose that there are constants $c_1, c_2, c_3, c_4, c_5 > 0$ so that, for all $t \in [0.T]$, $\| \bigtriangledown \dot{x}\|_\infty \le c_1$; $\| \bigtriangledown \cdot v\|_\infty \le c_2$; $\|v - \dot{x}\|_\infty \le c_3$; $\|\frac{Da}{Dt}\|_\infty, \| \bigtriangledown a\|_\infty$, $\|\bigtriangledown \frac{Da}{Dt}\|_\infty \le c_4$; and the norm of the jump in $\bigtriangledown \dot{x}$ across an edge $e = \bar{\Omega}_k \cap \bar{\Omega}_m$ is bounded by $c_5 \min\{h_k, h_m\}$. Then there is a constant $C(c_1, c_2, c_3, c_4, c_5, a_0, a_1, c, T, \Omega)$ such that*

$$
\begin{array}{rcl}
\|u - U\|(t) & \le & C\{\|h(\sum_i h_i^{2(\min\{r+1,s\}-1)} \|u\|^2_{H^s(\Omega_i)})^{1/2}\|_{L^\infty[0,T]} \\
& & +\|h(\sum_i h_i^{2(\min\{r+1,s\}-1)} \|\frac{Du}{Dt}\|^2_{H^s(\Omega_i)})^{1/2}\|_{L^2[0,T]}\},
\end{array} \tag{28}
$$

for any $t \in [0, T]$.

Proof: Again we will use Theorem 2 to establish this $L^2(\Omega)$ estimate. We will use an elliptic projection to give the Ψ that is in Theorem 2. The most tedious part of the proof is bounding the time derivative part of $\| \; \|_0$; we do that here by estimating the $L^2(\Omega)$-norm of that term.

Set $\mathcal{B}_1(\xi, \eta) = (a \bigtriangledown \xi, \bigtriangledown \eta)$, and define a linear projection $P : H^1_0(\Omega) \to \mathcal{M}(t)$ by

$$\mathcal{B}_1(\xi - P\xi, \phi) = 0, \; \forall \phi \in \mathcal{M}(t).$$

Denote $\eta = u - Pu$. For any given $t \in [0, T]$, let $\phi(x)$ be any function in $\mathcal{M}(t)$. Let $\psi(x, \tilde{t}) = \phi(\mathcal{G}(\mathcal{G}^{-1}(x, \tilde{t}), t))$ for any $\tilde{t} \in [0, T]$. It is easy to see that $\psi(x, t) = \phi(x)$ and $\frac{D\psi}{D\tilde{t}} = 0$ for any $\tilde{t} \in [0, T]$.

We have, at time t,

$$
\begin{array}{rcl}
0 & = & \frac{d}{dt}\{\mathcal{B}_1(\eta(., t), \psi(., t))\} = \sum_i\{\frac{d}{dt}\int_{\Omega_i} a \bigtriangledown \eta \cdot \bigtriangledown \psi dx\} \\
& = & \sum_i\{\int_{\Omega_i} a_t \bigtriangledown \eta \cdot \bigtriangledown \phi dx + \int_{\Omega_i} a \bigtriangledown \eta_t \cdot \bigtriangledown \phi dx \\
& & + \int_{\Omega_i} a \bigtriangledown \eta \cdot \bigtriangledown \psi_t dx + \int_{\partial\Omega_i} a \bigtriangledown \eta \cdot \bigtriangledown \phi(\dot{x} \cdot n) ds\},
\end{array} \tag{29}
$$

where n is the outer norm of $\partial\Omega_i$. Note that

$$\int_{\Omega_i} a \bigtriangledown \eta_t \cdot \bigtriangledown \phi dx = \int_{\Omega_i} a \bigtriangledown \frac{D\eta}{Dt} \cdot \bigtriangledown \phi dx - \int_{\Omega_i} a \bigtriangledown (\dot{x} \cdot \bigtriangledown \eta) \cdot \bigtriangledown \phi dx,$$

and

$$
\begin{array}{l}
\dot{x} \cdot \bigtriangledown(\bigtriangledown \eta \cdot \bigtriangledown \phi) = \sum\limits_k \dot{x}_k \partial_{x_k}(\sum\limits_j \partial_{x_j} \eta \partial_{x_j} \phi) \\
= \sum_k \sum_j (\dot{x}_k \partial_{x_j} \partial_{x_k} \eta \partial_{x_j} \phi + \dot{x}_k \partial_{x_j} \partial_{x_k} \phi \partial_{x_j} \eta) \\
= \bigtriangledown(\dot{x} \cdot \bigtriangledown \eta) \cdot \bigtriangledown \phi - (\bigtriangledown \eta)^T (\bigtriangledown \dot{x})(\bigtriangledown \phi) \\
\quad + \bigtriangledown(\dot{x} \cdot \bigtriangledown \phi) \cdot \bigtriangledown \eta - \bigtriangledown \phi)^T (\bigtriangledown \dot{x})(\bigtriangledown \eta).
\end{array} \tag{30}
$$

Using the fact that $0 = \frac{D\psi(x,t)}{Dt} = \psi_t(x,t) + \dot{x} \cdot \bigtriangledown\phi(x)$ we have

$$\begin{array}{l} \frac{d}{dt}\int_{\Omega_i} a \bigtriangledown \eta \cdot \bigtriangledown\psi dx \\ \quad = \int_{\Omega_i} \frac{Da}{Dt} \bigtriangledown \eta \cdot \bigtriangledown\phi dx + \int_{\Omega_i} a \bigtriangledown (\frac{D\eta}{Dt}) \cdot \bigtriangledown\phi dx + \int_{\Omega_i} a \bigtriangledown \eta \cdot \bigtriangledown\phi(\bigtriangledown \cdot \dot{x})dx \\ \qquad - \int_{\Omega_i} a(\bigtriangledown\eta)^T(\bigtriangledown\dot{x})(\bigtriangledown\phi)dx - \int_{\Omega_i} a(\bigtriangledown\phi)^T(\bigtriangledown\dot{x})(\bigtriangledown\eta)dx. \end{array} \tag{31}$$

Therefore we can write

$$\frac{d}{dt}\mathcal{B}_1(\eta,\psi) = \mathcal{B}_1(\frac{D\eta}{Dt},\phi) + E(\eta,\phi) = 0,$$

where

$$\begin{array}{rcl} E(\eta,\phi) & = & \int_\Omega \frac{Da}{Dt} \bigtriangledown \eta \cdot \bigtriangledown\phi dx + \int_\Omega a \bigtriangledown \eta \cdot \bigtriangledown\phi(\bigtriangledown \cdot \dot{x})dx \\ & & - \int_\Omega a(\bigtriangledown\eta)^T\{\bigtriangledown\dot{x} + (\bigtriangledown\dot{x})^T\}(\bigtriangledown\phi)dx. \end{array} \tag{32}$$

It is easy to see that $E(u,v) \le C\|u\|_1\|v\|_1$, so

$$\begin{array}{rcl} \|\frac{D\eta}{Dt}\|_1^2 & \le & C\mathcal{B}_1(\frac{D\eta}{Dt},\frac{D\eta}{Dt}) \\ & = & C\{\mathcal{B}_1(\frac{D\eta}{Dt},\frac{D\eta}{Dt}-\phi) + E(\eta,\frac{D\eta}{Dt}-\phi) - E(\eta,\frac{D\eta}{Dt})\} \\ & \le & C\{\|\frac{D\eta}{Dt}\|_1\|\frac{D\eta}{Dt}-\phi\|_1 + \|\eta\|_1\|\frac{D\eta}{Dt}-\phi\|_1 + \|\eta\|_1\|\frac{D\eta}{Dt}\|_1\}. \end{array} \tag{33}$$

It follows that

$$\|\frac{D\eta}{Dt}\|_1 \le C\{\|\eta\|_1 + \inf_{\phi\in\mathcal{M}(t)}\|\frac{Du}{Dt} - \phi\|_1\}. \tag{34}$$

Next we use a duality argument to get an estimate of $\|\frac{D\eta}{Dt}\|$. Let $\xi \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfy

$$\int_\Omega a \bigtriangledown \xi \cdot \bigtriangledown\zeta dx = \int_\Omega \frac{D\eta}{Dt}\zeta dx,\ \forall\zeta \in H_0^1(\Omega).$$

For any $\phi \in \mathcal{M}(t)$,

$$\begin{array}{rcl} (\frac{D\eta}{Dt},\frac{D\eta}{Dt}) & = & \mathcal{B}_1(\frac{D\eta}{Dt},\xi) \\ & = & \mathcal{B}_1(\frac{D\eta}{Dt},\xi-\phi) + E(\eta,\xi-\phi) - E(\eta,\xi), \end{array} \tag{35}$$

and by integration by parts,

$$\begin{array}{rcl} E(\eta,\xi) & = & -\int_\Omega \eta(\bigtriangledown\frac{Da}{Dt}\cdot\bigtriangledown\xi + \frac{Da}{Dt}\bigtriangleup\xi)dx - \sum_i\int_{\Omega_i}\eta\bigtriangleup\xi(\bigtriangledown\cdot\dot{x})dx \\ & & -\sum_j\int_{e_j}\eta\frac{\partial\xi}{\partial n_{e_j}}[\bigtriangledown\cdot\dot{x}]ds \\ & & +\sum_i\int_{\Omega_i}\eta\{(\bigtriangledown a)^T(\bigtriangledown\dot{x})(\bigtriangledown\xi) + a\bigtriangledown\cdot((\bigtriangledown\dot{x})(\bigtriangledown\xi))\}dx \\ & & +\sum_j\int_{e_j} a\,\eta([\bigtriangledown\dot{x}](\bigtriangledown\xi))\cdot n_{e_j}ds \\ & & +\sum_i\int_{\Omega_i}\eta\{(\bigtriangledown a)^T(\bigtriangledown\dot{x})^T(\bigtriangledown\xi) + a\bigtriangledown\cdot((\bigtriangledown\dot{x})^T(\bigtriangledown\xi))\}dx \\ & & +\sum_j\int_{e_j} a\,\eta([\bigtriangledown\dot{x}]^T(\bigtriangledown\xi))\cdot n_{e_j}ds. \end{array} \tag{36}$$

We need to take a close look at the integrals over the edges. Suppose that an edge $e = \bar{\Omega}_k \cap \bar{\Omega}_m$. Let $h(e) = \min\{h_k, h_m\}$. The first boundary integral in (36) can be bounded as follows:

$$
\begin{array}{l}
\sum_j \int_{e_j} \eta \frac{\partial \xi}{\partial n_{e_j}} [\nabla \cdot \dot{x}] ds \leq C \sum_j \|h^{1/2}(e_j)\eta\|_{L^2(e_j)} \|h^{1/2}(e_j) \frac{\partial \xi}{\partial n_{e_j}}\|_{L^2(e_j)} \\
\quad \leq C(\epsilon) \sum_j \|h^{1/2}(e_j)\eta\|^2_{L^2(e_j)} + \epsilon \sum_j \|h^{1/2}(e_j) \frac{\partial \xi}{\partial n_{e_j}}\|^2_{L^2(e_j)} \\
\quad \leq C(\epsilon) \sum_i (\|\eta\|^2_{L^2(\Omega_i)} + h_i^2 |\eta|^2_{H^1(\Omega_i)}) + C\epsilon \sum_i (|\xi|^2_{H^1(\Omega_i)} + h_i^2 |\xi|^2_{H^2(\Omega_i)}) \\
\quad \leq C(\epsilon)\{\|\eta\|^2 + \sum_i h_i^2 |\eta|^2_{H^1(\Omega_i)}\} + C\epsilon \|\frac{D\eta}{Dt}\|^2, \ \forall \epsilon > 0.
\end{array}
\tag{37}
$$

Similar results can be achieved for the other integrals over the edges e_j, so that by choosing ϵ small enough, we can conclude that

$$
|E(\eta, \xi)| \leq C\{\|\eta\|^2 + \sum_i h_i^2 |\eta|^2_{H^1(\Omega_i)}\} + \frac{1}{4} \|\frac{D\eta}{Dt}\|^2.
$$

Also choose $\phi \in \mathcal{M}(t)$ so that

$$
\mathcal{B}_1(\frac{D\eta}{Dt}, \xi - \phi) \leq C \|\frac{D\eta}{Dt}\|_1 \|\xi - \phi\|_1 \leq Ch \|\frac{D\eta}{Dt}\|_1 \|\frac{D\eta}{Dt}\|,
$$

and

$$
E(\eta, \xi - \phi) \leq Ch \|\eta\|_1 \|\frac{D\eta}{Dt}\|.
$$

Therefore we have from (35)

$$
\begin{array}{rcl}
\|\frac{D\eta}{Dt}\|^2 & \leq & C\{h^2 \|\frac{D\eta}{Dt}\|_1^2 + h^2 \|\eta\|_1^2 + \|\eta\|^2 + \sum_i h_i^2 |\eta|^2_{H^1(\Omega_i)}\} \\
& \leq & Ch^2 \sum_i h_i^{2(\min\{r+1,s\}-1)} \{\|\frac{Du}{Dt}\|^2_{H^s(\Omega_i)} + \|u\|^2_{H^s(\Omega_i)}\}.
\end{array}
\tag{38}
$$

The rest of the proof is an application of Theorem 2 using $\| \cdot \|_0$. Since $\|u - U\|_0$ dominates the term we want to bound, it suffices to show that $\|u - \Psi\|_0$ can be bounded by terms on the right-hand side of (25).

We choose $\Psi = Pu$. The estimate of $\|u - \Psi\|_{L^\infty(0,T;L^2(\Omega))}$ is straightforward. The weaker semi-norm on $\frac{D}{Dt}(u - \Psi)$ is also bounded from (38). The $H^1(\Omega)$-norm of $P_1(u - \Psi)$ can be bounded as follows. For any $\phi \in \mathcal{M}(t)$,

$$
\begin{array}{rl}
& \mathcal{B}(P_1(u - \Psi), \phi) = \mathcal{B}(u - \Psi, \phi) \\
= & \mathcal{B}_1(u - \Psi, \phi) + ((v - \dot{x}) \cdot \nabla(u - \Psi), \phi) + c_3(u - \Psi, \phi) \\
= & -((u - \Psi), \phi \nabla \cdot (v - \dot{x}) + (v - \dot{x}) \cdot \nabla \phi) + c_3(u - \Psi, \phi) \\
\leq & (\| \nabla \cdot (v - \dot{x})\|_{L^\infty(\Omega)} + c_3) \|u - \Psi\| \|\phi\| \\
& + \|v - \dot{x}\|_{L^\infty(\Omega)} \|u - \Psi\| \|\phi\|_1.
\end{array}
\tag{39}
$$

Using the coercivity of $\mathcal{B}(\cdot, \cdot)$ (see (11)) and taking $\phi = P_1(u - \Psi)$ we get that

$$\|P_1(u - \Psi)\|_1 \leq C\|u - \Psi\|. \quad \square$$

The $\frac{D\eta}{Dt}$ term was estimated in $L^2(\Omega)$ instead of the discrete H^{-1} semi-norm, so one might think that if (27) satisfies an H^3-regularity bound and $\dot{x}$ was smooth enough one might be able to weaken the norm on $\frac{D\eta}{Dt}$. We were not able to do this, except in trivial special cases.

6 Remarks

If we replace the boundary condition $w \cdot \nu = 0$ on Γ_N by $(w - \dot{x}) \cdot \nu = 0$ on Γ_N, Lemma 1 holds even if the domain Ω is time dependent. Therefore it seems possible to get analogous results in this situation. However, a more interesting situation is one in which mesh elements flow into and out of the domain, instead of just moving around in the domain; this will be the topic of future work.

References

[1] R. BANK & R. SANTOS, "Analysis of some moving space-time finite element methods," *SIAM J. Numer. Anal.*, v.30, 1993, pp. 1-18.

[2] J. DOUGLAS JR. & T. F. RUSSELL, "Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures," *SIAM J. Numer. Anal.*, v.19, 1982, pp. 871-885

[3] J. DOUGLAS Jr. & T. DUPONT, "Galerkin methods for parabolic equations," *SIAM J. Numer, Anal.*, v.7, 1970, pp. 575-626.

[4] T. DUPONT, "Mesh modification for evolution equations," *Math. Comp.*, v. 39, No. 159, July 1982, pp. 85-107.

[5] R. MILLER & K. MILLER, "Moving finite elements, part I," *SIAM J. Numer. Anal.*, v. 18, 1981, pp. 1019-1032.

[6] K. MILLER, "Moving finite elements, part II," *SIAM J. Numer. Anal.*, v.18, 1981, pp. 1033-1057.

[7] M. F. WHEELER, “A priori L^2 error estimates for nonlinear parabolic equations”, *SIAM J. Numer. Anal.*, v.10, 1973, pp. 723-759.